XH01 2012

PART A

1) Form $xyz - \lambda(ax+by+cz)$ & set first derivatives to 0

$yz = a\lambda,\ xz = b\lambda,\ xy = c\lambda \Rightarrow 3xyz = \lambda(ax+by+cz) = \lambda d \Rightarrow$

$\lambda = 3xyz/d$ & $x = d/3a,\ y = d/3b,\ z = d/3c$ & $\underline{xyz = d^3/27abc}$

(similar seen)

2) Use $f(x,y) = \sum_0^\infty \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^n \frac{f}{n!} = f(0,0) + (xf_x|_0 + yf_y|_0)$ $(f = \sin x\cos y)$

$+\frac{1}{2}(x^2 f_{xx}|_0 + 2xy f_{xy}|_0 + y^2 f_{yy}|_0) + \frac{1}{6}(x^3 f_{xxx}|_0 + 3x^2y f_{xxy}|_0 + 3xy^2 f_{xyy}|_0 + y^3 f_{yyy}|_0)$

$f(0,0) = 0$, $f_x = \underline{\cos x\cos y}$, $f_y = -\sin x\sin y$

$f_{xx} = -\sin x\cos y$, $f_{xy} = -\cos x\sin y$, $f_{yy} = -\sin x\cos y$

$f_{xxx} = -\underline{\cos x\cos y}$, $f_{xxy} = +\sin x\sin y$, $f_{xyy} = \underline{-\cos x\cos y}$, $f_{yyy} = \sin x\sin y$

& any sin will give zero

& $\sin x\cos y \approx \underline{x + \frac{1}{6}(-x^3 - 3xy^2)}$

3) EL $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial y}$

$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial y}.y' + \cancel{\frac{\partial F}{\partial y'}y''} - \cancel{y''\frac{\partial F}{\partial y'}} - y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \Rightarrow F - y'\frac{\partial F}{\partial y'} = \text{Const}$

4) χ-eqns are $\frac{dx}{18y} = \frac{dy}{-4x} = \frac{du}{xy/u}$

$\Rightarrow -4xdx = 18ydy$ & $9y^2 + 2x^2 = c_1$ & $\frac{dy}{-4} = \frac{u\,du}{y} \Rightarrow \frac{1}{2}y^2 + 2u^2 = c_2$

General solution is $\frac{1}{2}y^2 + 2u^2 = F(9y^2 + 2x^2)$. If $u(x,0) = x^2$ then,

$0 + 2x^4 = F(0 + 2x^2) \Rightarrow F(r) = 2r^2/4 = r^2/2$ & $2u^2 + \frac{1}{2}y^2 = \frac{1}{2}(9y^2 + 2x^2)^2$

$\Rightarrow \underline{u^2 = (x^2 + \frac{9}{2}y^2) - y^2/4}$.

5) $u = f(x+mt) \Rightarrow f'' = \frac{m^2}{c^2}f'' \Rightarrow m = \pm c$ & $u = f(x-ct) + g(x+ct)$

Putting $t=0 \Rightarrow F = f + g$ & $-f' + g' = G/c \Rightarrow -f + g = \frac{1}{c}\int_\alpha^x G(\xi)d\xi$

$\Rightarrow 2g = F + \frac{1}{c}\int_\alpha^x G(\xi)d\xi$, $2f = F - \frac{1}{c}\int_\alpha^x G(\xi)d\xi$. $u = f(x-ct) + g(x+ct)$

$= \frac{1}{2}(F(x-ct) + F(x+ct)) + \frac{1}{2c}\left(\int_\alpha^{x+ct} G(\xi)d\xi - \int_\alpha^{x-ct} G(\xi)d\xi\right)$ = required answer.

6) $u(x,t) = X(x)T(t) \Rightarrow \frac{1}{c^2}T''X = X''T \Rightarrow \frac{X''}{X} = \frac{T''}{c^2T} = \lambda$ a sep constant as l.h.s = function of x, r.h.s = " t (x & t independent)

$u(0,t) = u(\pi,t) = 0$ needs $\lambda < 0 = -p^2$ say & $X = A\sin px$ using

$X(0) = 0$, $X(\pi) = 0$ gives $p = n$, $n = 1,2,3\ldots$, $T'' + c^2n^2T = 0 \Rightarrow T = \cos(cnt)$ satisfying

$T'(0) = 0$ as $\partial u/\partial t|_0 = 0$. $\therefore u = \sum_1^\infty A_n\sin nx\cos(nct)$. At $t = 0$ we need $u = f =$

$\sum_1^\infty A_n\sin nx$. Using orthogonality of $\{\sin nx\}$ gives $A_n.\frac{\pi}{2} = \int_0^\pi f(x)\sin nx\,dx$, as required

Part B

7) a) Form. $F(x,y,y') = y'^2 - \lambda xy$ & EL $\Rightarrow \frac{d}{dx}\left[2y'\right] = -\lambda x \Rightarrow y' = -\frac{\lambda}{2}\left(x^2/2\right.$

$y = -\lambda/2\left(x^3/6 + ax + b\right)$, $y(0)=0 \Rightarrow b=0$, $y(1)=0 \Rightarrow a=-1/6$. $y = \lambda/12\left(x-x^3\right)$

$\int_0^1 xy\,dx = 1 = \frac{\lambda}{12}\left(\frac{1}{3}-\frac{1}{5}\right) \therefore \frac{\lambda}{12}\cdot\frac{2}{15}$, $\underline{y = \frac{15}{2}(x-x^3)}$, $y' = \frac{15}{2}(1-3x^2)$ &

$I[y] = \left(\frac{15}{2}\right)^2\left(1 - 6/3 + 9/5\right) = \frac{15}{4}(15-30+27) \therefore \frac{15}{4}\cdot 12 = \underline{45}$.

b) $F(x,y,y') = f(x,y)\sqrt{1+y'^2}$. EL $\Rightarrow \frac{d}{dx}\left[\frac{fy'}{\sqrt{1+y'^2}}\right] = f_y\sqrt{1+y'^2} \Rightarrow \frac{y''f}{\sqrt{\ }} - \frac{fy'y'y''}{(\ )^{3/2}}$

$+(f_x + y'f_y)\frac{y'}{\sqrt{\ }} = f_y\sqrt{\ }$. $\Rightarrow \frac{y''f}{\sqrt{\ }}\left(1 - \frac{y'^2}{1+y'^2}\right) + \frac{y'f_x}{\sqrt{\ }} + \frac{f_y}{\sqrt{\ }}\left(y'^2 - 1 - y'^2\right) = 0$.

$\underline{\Rightarrow \frac{y''f}{(1+y'^2)} + y'\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} = 0}$

If $f(x) = 1/x$, then $\partial f/\partial y = 0$ & $\frac{y''}{(1+y'^2)}\cdot\frac{1}{x} + y'\left(\frac{-1}{x^2}\right) = 0$. with $g = y'$ we get

$\frac{dg}{dx}\cdot\frac{1}{g(1+g^2)} = \frac{1}{x} = \frac{dg}{dx}\left[\frac{1}{g} - \frac{g}{1+g^2}\right] \Rightarrow \ln x = \ln g - \frac{1}{2}\ln(1+g^2) + \text{Const}$ &

$Ax = \frac{g}{\sqrt{1+g^2}} \Rightarrow \frac{1}{A^2x^2} = \frac{1}{g^2} + 1$, $g = \frac{dy}{dx} = \pm\frac{Ax}{\sqrt{1-A^2x^2}} \Rightarrow y + B = \pm\frac{1}{A}\sqrt{1-A^2x^2}$

$A^2(y+B)^2 + A^2x^2 = 1$

as required, centre $(0,-B)$ radius $1/A^2$

8) a) Parameterise initial data & χ-eqns

$\frac{dx}{dt} = x$, $\frac{dy}{dt} = y$, $\frac{du}{dt} = 2u$ with, at $t=0$, $x=s$, $y=1$, $u=s^2$

$\Rightarrow x = se^{-t}$, $y = e^{-t}$, $u = s^2e^{-2t}$ $\Rightarrow \underline{u = xy}$.

b) $\frac{\partial u}{\partial t} + \frac{1}{2}\left(\frac{\partial a}{\partial x}\cdot u + 2a\frac{\partial u}{\partial x}\right) = 0 \Rightarrow \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = -\frac{1}{2}\frac{\partial a}{\partial x}u$.

χ eqns are $dt = \frac{dx}{a} = \frac{du}{-\frac{1}{2}a'u} \Rightarrow t - \int^x\frac{d\xi}{a(\xi)} = C_1$ & $\frac{du}{u} = -\frac{1}{2}\frac{a'}{a}dx \Rightarrow \ln ua^{1/2} = C$.

$ua^{1/2} = C_2$

$\underline{\Rightarrow u = \frac{1}{a^{1/2}}G\left(t - \int^x\frac{d\xi}{a(\xi)}\right)}$ & if $u(0,t) = f(t)$, $\underline{u = \sqrt{\frac{a(0)}{a(x)}}f\left(t - \int_0^x\frac{d\xi}{a(\xi)}\right)}$

& if $a(x) = \frac{1}{1+x^2}$, $u(x,t) = \sqrt{\frac{1}{1+x^2}}f\left(t - \tan^{-1}x\right)$

(7)

q) Use separation of variables $\Theta = X(x)T(t)$, $X''T = XT' \mp hXT$, $\frac{X''}{X}$

a separation constant. $X'' - \lambda X = 0$, $T' = (\lambda + h)T$

If $T(0,t) = T(L,t) = 0$, then $X(0) = X(L) = 0$ & we need $\lambda < 0$, $\lambda = -p^2$ say

$X = A\sin px$ with $pL = n\pi$, $n = 1, 2, \ldots$ gives X, $*\ \theta = \sum_1^\infty A_n \sin\left(\frac{n\pi x}{L}\right) e^{\left(-\frac{n^2\pi^2}{L^2} + h\right)t}$

We need $\underline{h < \pi^2/L^2}$ so that $\theta \to 0$ as $t \to \infty$

a) If $T(0,t) = 0$, $T_x(0,t) = 0$ then $X = A\sin px$ & $\cos pL = 0 \Rightarrow p = (n + 1/2)\pi/L$

We need $\underline{h < \pi^2/4L^2}$, $\theta = \sum_0^\infty A_n \sin\left(n+\frac{1}{2}\right)\frac{\pi x}{L} e^{\left(-(n+1/2)^2\frac{\pi^2}{L^2} + h\right)t}$ *

c) Here $X'(0) = X'(L) = 0$ & $\lambda = 0$, $X = A_0$ is possible, as is $\lambda < 0$, $\lambda = -p^2$

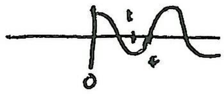
$X = A\cos px$ with $\sin pL = 0$, $p = \frac{n\pi}{L}$. If $\int_0^L f(x)dx = 0$ then $A_0 = 0$

& $\underline{h < \pi^2/L^2}$. If $\int_0^L f(x)dx \neq 0$ then $A \neq 0$ & $\underline{h < 0}$.

$\theta = e^{+ht}A_0 + \sum_1^\infty A_n \cos\frac{n\pi x}{L} e^{\left(-n^2\pi^2/L^2 + h\right)t}\ dt$ & $A_0\int_0^L e^{ht}dx + 0 = \int_0^L f(x)dx$.

With conditions as specified, use * & we need

$$1 = \sum_0^\infty A_n \sin\left(n+\frac{1}{2}\right)\frac{\pi x}{L} \quad \text{so} \quad \int_0^L \sin\left(n+\frac{1}{2}\right)\frac{\pi x}{L}dx = A_n \cdot \frac{L}{2}$$



$$\Rightarrow A_n = \frac{2}{L}\left[\frac{L}{\pi(n+1/2)}(-1)\cos\left(n+\frac{1}{2}\right)\frac{\pi x}{L}\right]_0^L = \frac{2}{\pi(n+1/2)}\left(1 - \underbrace{\cos(n+1/2)\pi}_{0}\right) = \frac{2}{\pi(n+1/2)}$$

$$\underline{\theta(x,t) = \frac{2e^{ht}}{\pi}\sum_0^\infty \frac{\sin(n+1/2)\pi x/L}{n+1/2} e^{-(n+1/2)^2\pi^2/L^2\, t}}$$